

4.1.4. Teskari matritsa

4.1.6-ta'rif. Agar bir xil tuzilishli A va B kvadrat matritsalar uchun $AB=BA=E$ (E -birlik matritsa) munosabat o'rinli bo'lsa, u vaqtda ularni *o'zaro teskari matritsalar* deyiladi.

Agar A kvadrat matritsa berilgan bo'lsa, unga teskari matritsani A^{-1} orqali belgilanadi. Yuqoridagi ta'rifga ko'ra

$$AA^{-1} = A^{-1}A = E$$

munosabat o'rinli bo'ladi.

Agar teskari matritsa mavjud bo'lsa, uning yagona bo'lishi ta'rifidan kelib chiqadi. Haqiqatdan ham, A matritsaga ikkita A_1^{-1} va A_2^{-1} teskari matritsalar mavjud deb faraz qilsak,

$$\begin{aligned} AA_1^{-1} &= A_1^{-1}A = E, \quad AA_2^{-1} = A_2^{-1}A = E \Rightarrow A_1^{-1}(AA_2^{-1}) = A_1^{-1}E \Rightarrow \\ &\Rightarrow (A_1^{-1}A)A_2^{-1} = A_1^{-1} \Rightarrow EA_2^{-1} = A_1^{-1} \Rightarrow A_2^{-1} = A_1^{-1} \end{aligned}$$

ni olamiz.

Berilgan kvadrat matritsaga teskari matritsa har doim ham mavjud bo'lavermaydi. Bu o'rinda quyidagi tasdiq to'g'ridir.

4.1.1-teorema. Matritsaga teskari matritsa mavjud bo'lishi uchun u maxsus bo'lmasligi (determinanti nolga teng bo'lmasligi) zarur va yetarlidir.

Isboti. $A=[a_{ij}]_n$ – kvadrat matritsa berilgan bo'lsin. Unga teskari matritsani $A^{-1}=[x_{ij}]_n$ deb faraz qilaylik. U vaqtda, $AA^{-1}=E$ tenglikdan

$$\sum_{k=1}^n a_{ik} \cdot x_{kj} = \delta_{ij}; \quad i = \overline{1;n}, \quad j = \overline{1;n} \quad (4.1.3)$$

dan iborat n noma'lumli n ta chiziqli tenglamalarning n ta sistemalarini olamiz. Bu sistemalar koeffitsiyentlari bir xil bo'lib, A matritsa elementlaridir, o'ng tomoni esa birlik matritsaning mos ustun elementlaridir (δ_{ij} – Kroneker belgisi ekanligini eslatamiz).

Demak, $\det A \neq 0$ bo'lsa, bu sistemalarning har biri yagona yechimga ega bo'lib, teskari matritsaning mos ustun elementlarini aniqlaydi. Bu teoremaning yetarli qismining isbotidir.

Endi, A^{-1} mavjud bo'lsin deylik, u vaqtda $AA^{-1}=E$ o'rinli bo'lib, bundan $\det A \cdot \det A^{-1} = 1$ ni olamiz. Agar $\det A = 0$ deb faraz qilsak, oxirgi tenglikdan $0=1$ ziddiyatli tenglikka kelamiz, demak, $\det A \neq 0$ bo'lar ekan. Bu teoremaning zaruriy qismining isbotidir.

Bu yerda Kramer formulalaridan foydalansak va (4.1.3) ning har bir sistemasining o'ng tomoni birlik matritsaning mos ustuni ekanligini e'tiborga olsak, $\det A \neq 0$ bo'lganda

$$x_{ij} = \frac{A_{ji}}{\det A}, \quad i = \overline{1;n}, \quad j = \overline{1;n}$$

ekanligi kelib chiqadi, bu yerda A_{ji} $\det A$ ning a_{ji} elementiga mos algebraik to'ldiruvchidir.

Agar $C = [A_{ji}]_n$ matritsani qarasak, u vaqtda

$$A^{-1} = \frac{1}{\det A} C^*$$

ekanligi ravshandir. Demak,

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \dots & \dots & \dots & \dots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{bmatrix}.$$

Ushbu

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}$$

matritsa uchun teskari matritsani topaylik. Buning uchun, avvalo, $\det A$ determinantni yozamiz va uni hisoblaymiz:

$$\det A = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 0 & -1 \\ 0 & -1 & -3 \end{vmatrix} = -1 \neq 0.$$

Demak, A – maxsusmas matritsa, unga teskari matritsa mavjud. Endi, $\det A$ ning har bir satr algebraik to'ldiruvchilarini hisoblaymiz:

$$\begin{aligned} A_{11} &= 4 \cdot 6 - 5 \cdot 5 = -1, & A_{12} &= -(2 \cdot 6 - 3 \cdot 5) = 3, & A_{13} &= 2 \cdot 5 - 3 \cdot 4 = -2, \\ A_{21} &= -(2 \cdot 6 - 5 \cdot 3) = 3, & A_{22} &= 1 \cdot 6 - 3 \cdot 3 = -3, & A_{23} &= -(5 \cdot 1 - 3 \cdot 2) = 1, \\ A_{31} &= 2 \cdot 4 - 4 \cdot 3 = -2, & A_{32} &= -(1 \cdot 5 - 3 \cdot 2) = 1, & A_{33} &= 1 \cdot 4 - 2 \cdot 2 = 0. \end{aligned}$$

Bularni mos ravishda ustunlar qilib yozib, C^* matritsani tuzamiz:

$$C^* = \begin{pmatrix} -1 & 3 & -2 \\ 3 & -3 & 1 \\ -2 & 1 & 0 \end{pmatrix}$$

Nihoyat, C^* ning barcha elementlarini $\det A = -1$ ga bo'lib teskari matritsaga ega bo'lamiz:

$$A^{-1} = \begin{pmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{pmatrix}.$$

Endi, $A \cdot A^{-1} = E$ ekanligini tekshiramiz. Haqiqatdan ham,

$$\begin{aligned} A \cdot A^{-1} &= \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix} \cdot \begin{pmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{pmatrix} = \\ &= \begin{pmatrix} 1 \cdot 1 + 2 \cdot (-3) + 3 \cdot 2 & 1 \cdot (-3) + 2 \cdot 3 + 3 \cdot (-1) & 1 \cdot 2 + 2 \cdot (-1) + 3 \cdot 0 \\ 2 \cdot 1 + 4 \cdot (-3) + 5 \cdot 2 & 2 \cdot (-3) + 4 \cdot 3 + 5 \cdot (-1) & 2 \cdot 2 + 4 \cdot (-1) + 5 \cdot 0 \\ 3 \cdot 1 + 5 \cdot (-3) + 6 \cdot 2 & 3 \cdot (-3) + 5 \cdot 3 + 6 \cdot (-1) & 3 \cdot 2 + 5 \cdot (-1) + 6 \cdot 0 \end{pmatrix} = \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = E \end{aligned}$$

Shunga o'xshash, $A^{-1} \cdot A = E$ ekani ham ko'rsatiladi.

4.1.5. Matritsaning rangi

Aytaylik, $A = [a_{ij}]_{m \times n}$ matritsa berilgan bo'lsin. Agar $k \leq \min(m, n)$ bo'lsa, A matritsaning k ta ustuni va k ta satri kesishishidagi elementlaridan hosil bo'lgan k -tartibli kvadrat matritsaning determinantini A matritsaning k - tartibli minori deb ataladi. Matritsaning har bir elementini uning birinchi tartibli minori deb qabul qilinadi.

Matritsaning rangi deb, uning noldan farqli minorlari tartiblarining eng kattasiga aytiladi.

Agar A matritsaning rangi r ga teng bo'lsa, bu matritsada hech bo'lmaganda bitta noldan farqli r -tartibli minor borligini, biroq, uning r dan katta tartibli minorlari mavjud bo'lsa, ularning barchasi nolga tengligini anglatadi. A matritsaning rangini $\text{rank } A$ yoki $r(A)$ orqali belgilanadi.

Ushbu matritsani qaraylik:

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 5 & 4 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

Uning yagona to'rtinchi tartibli minori nolga teng:

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 5 & 4 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{vmatrix} = 0$$

(ikkita satri bir xil bo'lgan determinant sifatida); uchinchi tartibli minorlaridan biri esa

noldan farqli, masalan, $\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 5 \\ 0 & -1 & 0 \end{vmatrix} = 5 \neq 0$. Demak, berilgan matritsaning rangi 3 ga teng,

ya'ni $r(A)=3$.

Matritsaning rangini uning ta'rifi bo'yicha topish uchun ko'p sondagi determinantlarni hisoblashga to'g'ri keladi. Bu ishni matritsadagi elementar almashtirishlar tushunchalari yordamida osonlashtirish mumkin.

Elementar almashtirishlar deb quyidagilarga aytiladi:

1) Matritsaning biror satri (ustuni) barcha elementlarini noldan farqli bir xil songa ko'paytirish yoki bo'lish;

2) Matritsaning biror satri (ustuni) barcha elementlariga boshqa satri (ustuni)ning mos elementlarini biror songa ko'paytirib qo'shish;

3) Matritsaning satrlari (ustunlari) o'rnini o'zaro almashtirish;

4) Matritsaning barcha elementlari nolga teng bo'lgan satrini (ustunini) tashlab yuborish.

Bir-biridan elementar almashtirishlar orqali hosil qilinadigan matritsalar *ekvivalent matritsalar* deb ataladi. Ekvivalent matritsalarining ranglari teng bo'lishi isbotlangandir.

Shuningdek, matritsada ko'pi bilan uning rangiga teng sondagi chiziqli erkli satrlari (ustunlari) mavjud bo'lib, ular matritsa rangiga teng tartibli noldan farqli minoriga mos keluvchi satrlaridan (ustunlaridan) iborat bo'lishi isbotlangandir. Agar matritsaning rangiga teng sondagi uning chiziqli erkli satrlari (ustunlari) sistemasi aniqlangan bo'lsa, ularning chiziqli kombinatsiyasi orqali qolgan satrlarini (ustunlarini) ifodalash mumkin bo'ladi va elementar almashtirishlar yordamida ularga mos satrlarining

(ustunlarining) elementlari nollardan iborat bo'lgan ekvivalent matritsani olish mumkin [4]. Bu aytilganlarni matritsaning rangini topish jarayoniga qo'llash ishni birmuncha osonlashtiradi.

Masalan,

$$A = \begin{pmatrix} 3 & 0 & 1 & -3 & -2 \\ 3 & 4 & 2 & -1 & -1 \\ 6 & 4 & 3 & -4 & -3 \end{pmatrix}$$

matritsaning rangini hisoblaylik. Berilgan matritsaning birinchi satri elementlarini -1 ga ko'paytirib, uchinchi satriga qo'shaylik:

$$A_1 = \begin{pmatrix} 3 & 0 & 1 & -3 & -2 \\ 3 & 4 & 2 & -1 & -1 \\ 3 & 4 & 2 & -1 & -1 \end{pmatrix}$$

endi, A_1 ning 2-satrini -1 ga ko'paytirib, uchinchi satriga qo'shsak,

$$A_2 = \begin{pmatrix} 3 & 0 & 1 & -3 & -2 \\ 3 & 4 & 2 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

ni olamiz.

A_2 matritsaning nollardan iborat uchinchi satrini tashlab yuborib,

$$A_3 = \begin{pmatrix} 3 & 0 & 1 & -3 & -2 \\ 3 & 4 & 2 & -1 & -1 \end{pmatrix}$$

matritsaga kelamiz; uning rangi ikkiga tengligi ravshandir. Demak, berilgan matritsaning rangi ham ikkiga teng, ya'ni $r(A)=2$.

4.2. Chiziqli algebraik tenglamalar sistemasini matritsalar yordamida tekshirish

4.2.1. Chiziqli algebraik tenglamalar sistemasini matritsalar vositasida yozish va yechish

Ushbu tenglamalar sistemasi berilgan bo'lsin:

$$\sum_{j=1}^n a_{ij} x_j = c_i, \quad i = \overline{1; m}. \quad (4.2.1)$$

Bu sistemaning koeffitsiyentlaridan tuzilgan matritsani hamda noma'lumlari va ozod hadlarini matritsa-ustunlar (ustun-vektorlar) sifatida yozamiz:

$$A = [a_{ij}]_{m \times n}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix}, \quad C = \begin{pmatrix} c_1 \\ c_2 \\ \dots \\ c_m \end{pmatrix}. \quad (4.2.2)$$

A - sistemaning matritsasi X -noma'lum vektor, C esa sistemaning o'ng tomoni deb yuritiladi.

Bizga ma'lum bo'lgan matritsalarini ko'paytirish qoidasidan va matritsalarining tengligi shartidan foydalanib, (4.2.1) tenglamani (4.2.2) asosida, quyidagicha yozish mumkin:

$$AX=C. \quad (4.2.3)$$

(4.2.3) ni *matritsaviy tenglama* deb ataladi.

Agar (4.2.1) sistema (4.2.3) matritsaviy shaklda yozilgan bo'lib, $m=n$ hamda sistemaning A matritsasi maxsus bo'lmasa, bu tenglama quyidagicha yechiladi. Uning har ikki tomonini chapdan A^{-1} matritsaga ko'paytiramiz:

$$A^{-1}(AX)=A^{-1} \cdot C.$$

Matritsalarini ko'paytirishning guruhlash qonunidan foydalanib, oxirgini quyidagicha yozish mumkin:

$$(A^{-1}A) \cdot X=A^{-1} \cdot C.$$

$A^{-1}A=E$ va $EX=X$ bo'lgani uchun (4.2.3) matritsaviy tenglamaning yechimini ushbu ko'rinishda hosil qilamiz:

$$X=A^{-1} \cdot C. \quad (4.2.4)$$

Masalan,

$$\begin{cases} x_1 + 2x_2 = 10, \\ 3x_1 + 2x_2 + x_3 = 23, \\ x_2 + 2x_3 = 13 \end{cases}$$

tenglamalar sistemasini matritsaviy usulda yechaylik.

Buning uchun

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}; \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}; \quad C = \begin{pmatrix} 10 \\ 23 \\ 13 \end{pmatrix}$$

matritsalarini kiritamiz va teskari matritsani topish uchun quyidagilarni bajaramiz:

$$\det A = \begin{vmatrix} 1 & 2 & 0 \\ 3 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix} = 4 + 0 + 0 - 0 - 12 - 1 = -9 \neq 0.$$

A maxsusmas ekan, demak, unga teskari matritsa mavjud. Endi, A_{ij} algebraik to'ldiruvchilarni hisoblaymiz:

$$\begin{aligned} A_{11} &= \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 4 - 1 = 3; & A_{12} &= -\begin{vmatrix} 3 & 1 \\ 0 & 2 \end{vmatrix} = -6; & A_{13} &= \begin{vmatrix} 3 & 2 \\ 0 & 1 \end{vmatrix} = 3; \\ A_{21} &= -\begin{vmatrix} 2 & 0 \\ 1 & 2 \end{vmatrix} = -4; & A_{22} &= \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} = 2; & A_{23} &= -\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = -1; \\ A_{31} &= \begin{vmatrix} 2 & 0 \\ 2 & 1 \end{vmatrix} = 2; & A_{32} &= -\begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix} = -1; & A_{33} &= \begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} = -4. \end{aligned}$$

Teskari matritsani hisoblaymiz:

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix} = \frac{1}{-9} \begin{pmatrix} 3 & -4 & 2 \\ -6 & 2 & -1 \\ 3 & -1 & -4 \end{pmatrix},$$

$$A^{-1} = \begin{pmatrix} -1/3 & 4/9 & -2/9 \\ 2/3 & -2/9 & 1/9 \\ -1/3 & 1/9 & 4/9 \end{pmatrix}.$$

Berilgan sistemaning yechimini (4.2.4) ko‘rinishda yozamiz:

$$X = A^{-1} \cdot C = \begin{pmatrix} -1/3 & 4/9 & -2/9 \\ 2/3 & -2/9 & 1/9 \\ -1/3 & 1/9 & 4/9 \end{pmatrix} \cdot \begin{pmatrix} 10 \\ 23 \\ 13 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix}.$$

Bu yerdan, matritsalarining tenglik shartiga asosan, $x_1=4$, $x_2=3$, $x_3=5$ kelib chiqadi.